Bayesian Growth Curve Modeling with Measurement Error in Time

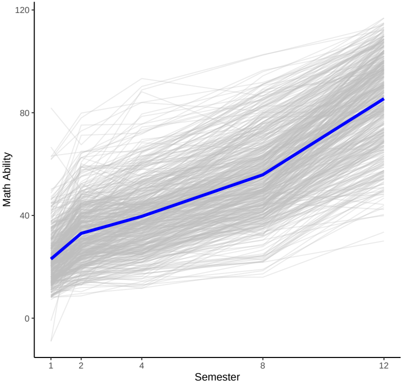
Lijin Zhang[1], Wen Qu[2], Zhiyong Zhang[3]

[1] Stanford University
[2] Fudan University
[3] University of Notre Dame

August 31, 2023

Growth Curve

- Tracking changes over time is vital for understanding the nature of development in abilities, personality, behavioral problems, and more.



Growth Curve Modeling (GCM)

- GCM is a powerful approach for tracing and describing patterns of change over time.
- A beauty of GCM lies in its ability to encapsulate both individual change and population trends.
- Linear Growth Curve Model

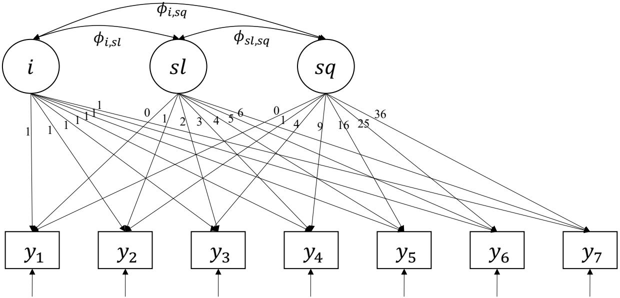
$$y_{nt} = i_n + sl_n \cdot (t - k) + \epsilon_{nt}, \quad n = 1, \ldots, N, t = 1, \ldots, T \quad (1)$$

  - y_{nt}: Response of individual n at time t
  - i_n: Intercept for the individual n
  - sl_n: Linear slope
  - ϵ_{nt}: Measurement error of y_{nt}, $\sim N(0, \psi_t)$

Quadratic Growth Curve Model

- If a linear growth curve does not fit well and a non-linear trend emerges from the longitudinal plot, researchers might opt for the quadratic growth model:

$$y_{nt} = i_n + sl_n \cdot (t - k) + sq_n \cdot (t - k)^2 + \epsilon_{nt} \qquad (2)$$



Quadratic Growth Curve Model

$$y_{nt} = i_n + sl_n \cdot (t - k) + sq_n \cdot (t - k)^2 + \epsilon_{nt} \tag{3}$$

If we assume $t = 1, 2, \ldots, 7$ and $k = 1$, the loading matrix $\mathbf{\Lambda}$ linking the latent and observed variables:

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix} \tag{4}$$

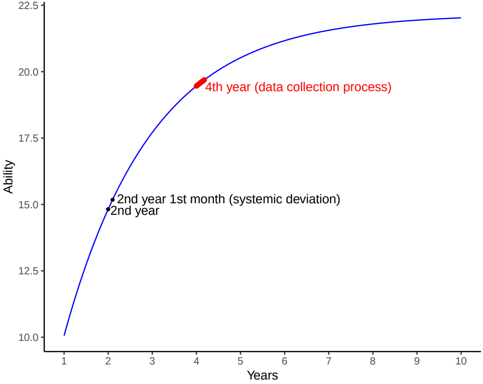
Time Intervals

- A growth curve model does not require the measurement to be equally spaced.
- Consider the scenario where measurements are taken in the 1st, 3rd, 6th, and 7th years.

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \tag{5}$$

Assumption

- Measurements should be conducted strictly at pre-set time or intervals.
- Specifically, the measurement time for each participant should be exactly maintained as designed.



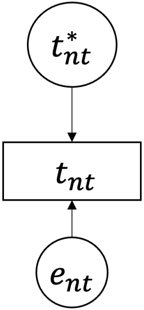
Measurement Error in Time

- This error can be broadly classified into two types: systematic and random.
- Systematic errors could be attributed to minor deviations from an ideal measurement schedule.
- Random errors might arise from the duration of the data collection process.
- Very common in real-world analysis, but the traditional model does not account for this error, assuming that all measurements strictly adhere to the pre-defined interval.

Aims

- Investigate the consequences of ignoring the measurement error in time.
- Develop a model that can integrate prior knowledge to handle error in time.

Modeling Error in Time



- t_{nt} denotes the recorded time value for the n-th individual at the t-th time point.
- t^*_{nt} is the true time value.
- e_{nt} is the measurement error in time.

Known t^*_{nt}

When the individual time values are known ($t_{nt} = t^*_{nt}$), t^*_{nt} can be directly included into the model as a regular model such as:

$$y_{nt} = i_n + sl_n \cdot (t_{nt} - k) + sq_n \cdot (t_{nt} - k)^2 + \epsilon_{nt} \tag{6}$$

The individualized loading matrix $\mathbf{\Lambda}_n$:

$$\mathbf{\Lambda}_n = \begin{bmatrix} 1 & t_{n1} - k & (t_{n1} - k)^2 \\ 1 & \dots & \dots \\ 1 & t_{nt} - k & (t_{nt} - k)^2 \\ 1 & \dots & \dots \\ 1 & t_{nT} - k & (t_{nT} - k)^2 \end{bmatrix} \tag{7}$$

Unknown t^*_{nt}

When the true individual time information is unavailable, we proposed to model t^*_{nt} as an unobserved variable.

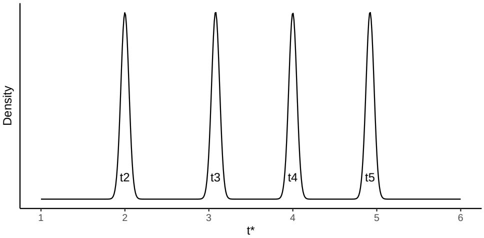
We anchor k and t^*_{1n} at 1 to help interpret the intercept parameter as the initial status for each individual.

We use Bayesian estimation to assign priors to t^*_{nt} starting from the second time point, which are centered around μ_t:

$$
\begin{aligned}
&y_{nt} = i_n + sl_n \cdot (t^*_{nt} - 1) + sq_n \cdot (t^*_{nt} - 1)^2 + \epsilon_{nt} \\
&t^*_{nt} \sim TN(\mu_t, \tau_r^2, t - 0.99, t + 0.99), \mu_t \sim N(t, \tau_s^2), \text{for } t = 2, \ldots, T \\
&\epsilon_{nt} \sim N(0, \psi_t) \\
&X_n \sim \text{MVN}\left(\begin{bmatrix} \mu_i & \mu_{sl} & \mu_{sq} \end{bmatrix}^T, \Phi\right), X_n = \begin{bmatrix} i_n & sl_n & sq_n \end{bmatrix}^T
\end{aligned}
\tag{8}
$$

Modeling Error in Time

- This approach incorporates the prior knowledge about the measurement schedule and addresses the error in time by modeling t^*_{nt}.
- The time t^*_{nt} is approximately fixed at t but allows slight variations.
- This greatly simplifies the interpretation of the slopes.



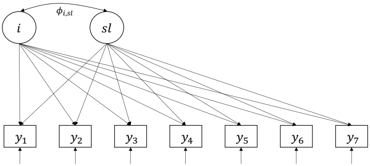
Simulation Studies

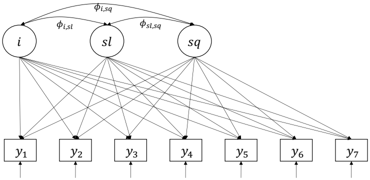
- Simulation 1: Consequences of Ignoring the Measurement Error in Time
- Simulation 2: Growth Curve Modeling with Known t^*_{nt}
- Simulation 3: Growth Curve Modeling with Unknown t^*_{nt}

Simulation Study 1

Consequences of Ignoring the Measurement Error in Time.

True Model:





Design

- The means of time ($\boldsymbol{\mu}_t$). Two scenarios were considered: $\boldsymbol{\mu}_{t0} = (1, 2, 3, 4, 5, 6, 7)$, and an alternate sequence $\boldsymbol{\mu}_{t1} = (1, 2.1, 2.9, 4.1, 4.9, 6.1, 6.9)$ with systemic derivation from $\boldsymbol{\mu}_{t0}$.
- The standard deviations (τ_r): 0, 0.1, 0.2, 0.3.
- Latent means for the intercept and slopes were set at 1 and 0.2 / 1 and 0.5, respectively.
- The residual variances for y_{nt} were established as either 1 or 4.
- Sample size: 250 and 500.

We simulated each condition of the combined factors 100 times.

Model Estimation

For the latent means, the variance-covariance matrix of the latent variables, and the residual variances of observed variables, we assigned either diffuse or weakly informative priors.

For example, priors for the quadratic growth curve model:

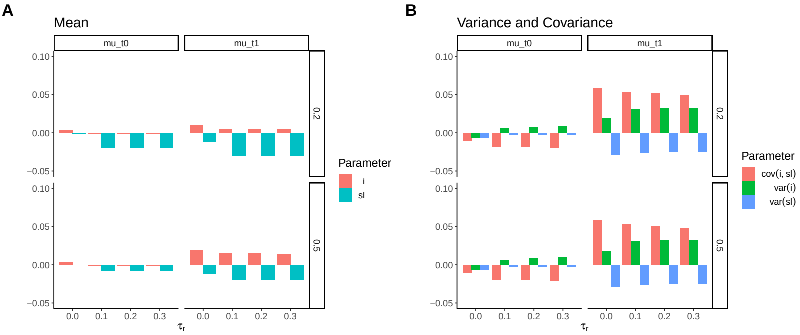
$$\begin{aligned}
&\begin{bmatrix} i_n \\ sl_n \\ sq_n \end{bmatrix} \sim MVN\left(\begin{bmatrix} \mu_i \\ \mu_{sl} \\ \mu_{sq} \end{bmatrix}, \Phi\right) \\
&\mu_i, \mu_{sl}, \mu_{sq} \sim N(0, 10) \\
&\Phi^{-1} \sim Wishart(I, 3) \\
&\psi_t^{-1} \sim Gamma(1, 1)
\end{aligned} \tag{9}$$

Model Estimation

- Software: JAGS
- Two MCMC chains were generated for convergence check and model estimation.
- Burn-in phase: 5,000 - 100,000 iterations.
- If the model converged within 100,000 iterations (EPSR $<$ 1.1), 5,000 more iterations would be generated for estimation.

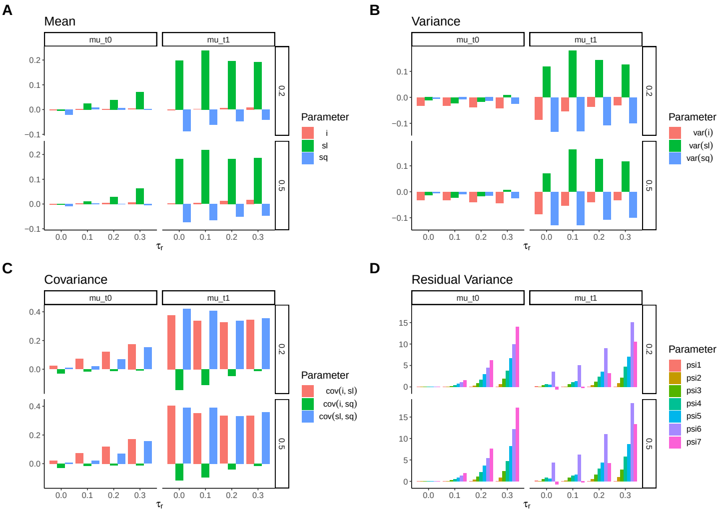
Results: High convergence rate ($>$95%) and power for latent means ($>$90%) across all modeling conditions.

Linear Growth Curve - Relative Bias



Slight increase in RB when time-based errors are present. However, these values still fall within an acceptable range (10%).

Quadratic Growth Curve - Relative Bias



Summary

- The measurement error in time seems to be captured by the residual error ϵ_{nt} of y_{nt} in the linear growth curve model, thereby having little influence on other parameter estimates.

$$y_{nt} = i_n + sl_n \cdot (t - k) + \epsilon_{nt} \tag{10}$$

- As τ_r increased under the $\boldsymbol{\mu}_{t0}$ conditions, the relative bias of $\mu_{sl}, \phi_{i,sl}, and \phi_{sl,sq}$ increased.
- When there were systematic deviations from the measurement schedule, considerable bias emerged (with $|RB| > 0.1$) for almost every parameter, except for μ_i.

Simulation Study 2

Models Comparison: Fixed t (traditional model) vs Known t^*_{nt}

- In practical scenarios, responses can be precisely timed, often through online surveys.
- We selected two conditions from the simulation study 1 where $N = 500$, $\tau_r = 0.2$ and $\mu_{sl} = \mu_{sq} = 0.2$.
- We varied the average response times: $\boldsymbol{\mu}_t = \boldsymbol{\mu}_{t0}$ or $\boldsymbol{\mu}_{t1}$.
- We saved the true individual time values from data generation for subsequent estimation.

Results

Table 2: Results of the Simulation Study 2.

	Model	t				Known t^*_{nt}			
		$\boldsymbol{\mu}_{t0}$		$\boldsymbol{\mu}_{t1}$		$\boldsymbol{\mu}_{t0}$		$\boldsymbol{\mu}_{t1}$	
	TRUE	RB(%)	RMSE	RB	RMSE	RB	RMSE	RB	RMSE
μ_i	1	0.27	0.06	0.59	0.06	0.26	0.06	0.27	0.06
μ_{sl}	0.2	3.92	0.06	**19.57**	0.07	1.96	0.05	1.94	0.05
μ_{sq}	0.2	0.64	0.05	-4.70	0.04	0.82	0.05	0.82	0.05
ϕ_i	1	-3.82	0.16	-3.66	0.15	-2.23	0.13	-2.68	0.13
ϕ_{sl}	1	-1.68	0.14	**14.27**	0.22	-1.70	0.11	-1.95	0.11
ϕ_{sq}	1	-1.40	0.07	**-10.81**	0.12	-0.58	0.07	-0.58	0.07
$\phi_{i,sl}$	0.4	**11.97**	0.11	**32.54**	0.17	3.72	0.08	4.50	0.09
$\phi_{i,sq}$	0.4	-1.21	0.07	-4.57	0.07	-1.34	0.07	-1.43	0.07
$\phi_{sl,sq}$	0.4	6.90	0.07	**33.66**	0.15	0.12	0.05	0.18	0.05
ψ_1	1	6.23	0.15	7.31	0.16	3.98	0.12	4.47	0.13
ψ_2	1	**28.95**	0.31	**38.58**	0.41	0.54	0.07	0.52	0.07
ψ_3	1	**85.23**	0.88	**122.30**	1.25	-0.34	0.09	-0.38	0.09
ψ_4	1	**170.87**	1.74	**243.23**	2.47	2.15	0.08	2.25	0.08
ψ_5	1	**298.60**	3.03	**353.47**	3.58	-0.32	0.09	-0.34	0.09
ψ_6	1	**445.47**	4.53	**906.59**	9.15	-0.03	0.12	-0.22	0.13
ψ_7	1	**623.08**	6.44	**321.02**	3.74	3.33	0.26	3.44	0.25

Simulation Study 3

- We varied two factors in data generation, including:
  1. The average response times, denoted as $\boldsymbol{\mu}_t$, were set to either $\boldsymbol{\mu}_{t0} = (1, 2, 3, 4, 5, 6, 7)$ or $\boldsymbol{\mu}_{t1} = (1, 2.1, 2.9, 4.1, 4.9, 6.1, 6.9)$;
  2. The standard deviation, τ_r, was set to either 0 or 0.2.
- All other settings were consistent with those of study 2 (e.g., $\mu_{sl} = \mu_{sq} = 0.2$, $N = 500$).
- Model comparison: Fixed t vs Unknown t^*_{nt}
- For the parameters τ_r and τ_s, we utilized hyper-priors to derive their values:

$$t^*_{nt} \sim TN(\mu_t, \tau_r^2, t{-}0.99, t{+}0.99), \mu_t \sim N(t, \tau_s^2), \text{for } t = 2, 3, \ldots, 7$$

$$\tau_r^{-2} \sim U(100, 10000), \tau_s^{-2} \sim U(1000, 10000)(11)$$

Results

Table 3: Results of the Simulation Study 3.

	t								Unknown t^*_{nt}							
	$\boldsymbol{\mu}_{t0}$				$\boldsymbol{\mu}_{t1}$				$\boldsymbol{\mu}_{t0}$				$\boldsymbol{\mu}_{t1}$			
	$\tau_r = 0$		$\tau_r = 0.2$		$\tau_r = 0$		$\tau_r = 0.2$		$\tau_r = 0$		$\tau_r = 0.2$		$\tau_r = 0$		$\tau_r = 0.2$	
	RB(%)	RMSE	RB	RMSE	RB	RMSE	RB	RMSE	RB	RMSE	RB	RMSE	RB	RMSE	RB	RMSE
μ_i	-0.29	0.07	0.27	0.06	-0.34	0.07	0.59	0.06	-0.24	0.07	0.41	0.06	0.38	0.07	0.64	0.06
μ_{sl}	-0.49	0.06	3.92	0.06	**19.81**	0.07	**19.57**	0.07	-0.98	0.06	-2.45	0.06	7.81	0.06	-0.99	0.06
μ_{sq}	-2.13	0.05	0.64	0.05	-8.88	0.05	-4.70	0.04	-1.84	0.05	2.37	0.05	-5.24	0.05	1.67	0.05
ϕ_i	-3.15	0.13	-3.82	0.16	-8.69	0.15	-3.66	0.15	-4.39	0.14	-4.08	0.16	-4.52	0.14	-4.28	0.16
ϕ_{sl}	-1.20	0.10	-1.68	0.14	**11.78**	0.16	**14.27**	0.22	-2.06	0.10	-2.47	0.14	6.10	0.13	-1.87	0.14
ϕ_{sq}	-0.48	0.07	-1.40	0.07	**-13.17**	0.14	**-10.81**	0.12	-0.32	0.06	1.72	0.07	-5.27	0.08	0.74	0.07
$\phi_{i,sl}$	2.23	0.09	**11.97**	0.11	**37.69**	0.17	**32.54**	0.17	4.18	0.09	5.13	0.10	**17.37**	0.12	**10.36**	0.11
$\phi_{i,sq}$	-2.96	0.06	-1.21	0.07	**-14.32**	0.08	-4.57	0.07	-2.76	0.07	1.64	0.08	-3.87	0.07	3.21	0.08
$\phi_{sl,sq}$	0.86	0.06	6.90	0.07	**42.13**	0.18	**33.66**	0.15	0.07	0.06	-8.63	0.07	**16.69**	0.10	-6.42	0.06
ψ_1	5.13	0.14	6.23	0.15	**12.96**	0.18	7.31	0.16	6.23	0.15	5.90	0.15	8.05	0.16	7.35	0.16
ψ_2	1.39	0.08	**28.95**	0.31	0.81	0.08	**38.58**	0.41	1.19	0.08	**18.67**	0.21	1.42	0.08	**22.32**	0.25
ψ_3	-0.99	0.08	**85.23**	0.88	**46.04**	0.48	**122.30**	1.25	-1.46	0.08	**43.72**	0.47	-1.32	0.08	**40.54**	0.43
ψ_4	1.73	0.10	**170.87**	1.74	**69.10**	0.71	**243.23**	2.47	1.03	0.10	**78.26**	0.81	0.83	0.10	**83.02**	0.85
ψ_5	1.10	0.09	**298.60**	3.03	**54.79**	0.57	**353.47**	3.58	-0.17	0.09	**120.32**	1.24	-0.38	0.09	**117.29**	1.21
ψ_6	-0.48	0.12	**445.47**	4.53	**354.77**	3.56	**906.59**	9.15	-3.04	0.12	**156.26**	1.63	-1.78	0.13	**164.81**	1.73
ψ_7	4.54	0.27	**623.08**	6.44	**-66.94**	0.67	**321.02**	3.74	3.09	0.27	**227.81**	2.50	-0.41	0.24	**216.21**	2.37

Model Selection

- We further evaluated how the DIC (Deviance Information Criterion) differentiates between the two models.
- For every replication, we used the DIC to identify the model with a better fit, as indicated by a lower DIC value.

Table: Model Selection Rates between Model t and Model t^*_{nt} using DIC.

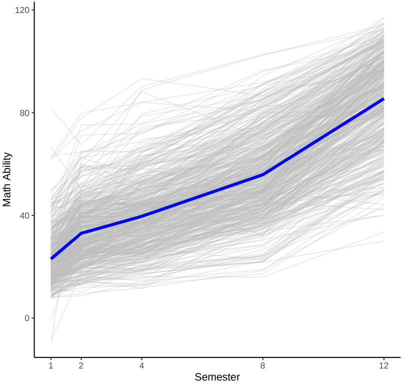
	$\boldsymbol{\mu}_{t0}$		$\boldsymbol{\mu}_{t1}$	
	t	t^*_{nt}	t	t^*_{nt}
$\tau_r = 0$	0.59	0.41	0	1
$\tau_r = 0.2$	1	0	0.17	0.83

Empirical Data

- Early Childhood Longitudinal Study—Kindergarten (ECLS-K).
- We extracted 500 samples of the math IRT (Item Response Theory) scale scores from five waves of the ECLS-K: the fall of kindergarten, and the spring of kindergarten, 1st, 3rd, and 5th grades.
- The ECLS-K study extended over many years in different US locations.
- This makes it challenging to ensure consistent measurement intervals for each individual, which could result in the measurement error in time.

Trajectory Plot

- We set the time interval unit as one semester.
- Five waves: the first, second, fourth, eighth, and twelfth semesters.



Model Comparison

- The first model had a fixed loading matrix:

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 7 & 49 \\ 1 & 11 & 121 \end{bmatrix} \tag{12}$$

- The second model integrated the unknown t^*_{nt} and allowed slight deviations from the designed measurement schedule:

$$\begin{aligned} & t = 2, t^*_{n2} \sim TN(\mu_2, \tau_r^2, 1.01, 2.99), \mu_2 \sim N(2, \tau_s^2) \\ & t = 4, t^*_{n4} \sim TN(\mu_4, \tau_r^2, 3.01, 4.99), \mu_4 \sim N(4, \tau_s^2) \\ & t = 8, t^*_{n8} \sim TN(\mu_8, \tau_r^2, 7.01, 8.99), \mu_8 \sim N(8, \tau_s^2) \\ & t = 12, t^*_{n,12} \sim TN(\mu_{12}, \tau_r^2, 11.01, 12.99), \mu_{12} \sim N(12, \tau_s^2) \\ & \tau_r^{-2}, \tau_s^{-2} \sim U(1000, 10000) \end{aligned} \tag{13}$$

Priors

- Echoing the simulation study, we utilized diffuse or weakly informative priors for other parameters in both models:

$$\begin{aligned} \mu_i &\sim N(20, 100), & \mu_{sl}, \mu_{sq} &\sim N(0, 100) \\ \Phi^{-1} &\sim Wishart(I, 3), & \psi_t^{-1} &\sim Gamma(1, 1) \end{aligned} \tag{14}$$

- Two MCMC chains were generated, with 50,000 iterations for burn-in and another 50,000 for inference. Both models reached convergence within the burn-in phase (EPSR < 1.1).

Results

Table 5: Results of the Empirical Study.

Model	t			t^*_{nt}		
Parameter	Estimate	SD	HPD	Estimate	SD	HPD
μ_i	26.182	0.997	(25.157, 27.352)	25.499	0.507	(24.484, 26.467)
μ_{sl}	3.896	0.354	(3.477, 4.254)	4.155	0.176	(3.807, 4.498)
μ_{sq}	0.135	0.028	(0.105, 0.168)	0.116	0.014	(0.087, 0.144)
ϕ_i	82.684	7.765	(69.348, 97.082)	81.426	6.684	(68.713, 94.82)
ϕ_{sl}	2.986	0.746	(1.938, 4.017)	2.935	0.503	(1.96, 3.916)
ϕ_{sq}	0.027	0.005	(0.02, 0.035)	0.027	0.003	(0.02, 0.034)
$\phi_{i,sl}$	14.711	1.479	(11.911, 17.5)	14.402	1.352	(11.801, 17.085)
$\phi_{i,sq}$	-1.035	0.127	(-1.282, -0.797)	-1.001	0.116	(-1.234, -0.778)
$\phi_{sl,sq}$	-0.217	0.058	(-0.301, -0.135)	-0.210	0.040	(-0.29, -0.134)
ψ_1	43.148	12.286	(34.487, 51.072)	36.870	3.397	(30.521, 43.783)
ψ_2	33.717	19.797	(26.746, 39.244)	29.814	2.846	(24.372, 35.44)
ψ_3	22.386	17.344	(16.655, 27.691)	23.345	2.709	(18.164, 28.771)
ψ_4	83.617	11.534	(70.738, 95.98)	76.804	5.984	(65.249, 88.563)
ψ_5	2.051	10.116	(0.119, 6.507)	1.969	2.293	(0.129, 6.335)
τ_r	-	-	-	0.015	0.005	(0.01, 0.025)
τ_s	-	-	-	0.031	0.001	(0.03, 0.032)
DIC		139286.85			118853.67	

Takeaways

- Ignoring the measurement error in time can lead to biased results in quadratic growth curve modeling.
- The proposed model introduces underlying individual time values that exist behind the preset measurement schedule.
- It outperforms the traditional model that ignores time errors in terms of estimation accuracy.
- Even in the absence of time errors, this model continues to provide excellent performance with acceptable bias.

Thanks for Listening!

Slides:
`https://lijinzhang.com/share/230831_gcm.pdf`